Horizontal diffusion operators in models with a stretched horizontal geometry

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1 Introduction

In stretched horizontal geometries, the definition of a convenient horizontal diffusion (HD) operator is not straightforward. In this note, we present the rationale that we used to define what is the "ideal" form of a HD operator when a general stretched geometry is used; then we show that this "target" operator has a very simple form in the particular geometry of the conformal Schmidt transformation, even when coupled to the use of the spectral (Fourier-Legendre) method on the sphere, as it is the case in ARPEGE. Compared to the original HD operator (Yessad and Bénard, 1995, YB95 herefater), this new HD operator has multiple advantages: besides its increased simplicity and cost-effectiveness, it does not suffer of any approximation, and it is much easier to tune since the total number of degrees of freedom can be brought down to two (the "global strength" and the "order" of the diffusion) for each variable to be diffused. If the second part of this work (devoted to the Schmidt transformation) is of rather particular interest, the first part (which deals with the definition of an "ideal" HD operator for stretched geometries) is of general interest for any stretched model.

2 HD operators for non-stretched geometries

The rationale to define an ideal HD operator for stretched geometries lies on the empirical observation of the way the strength of HD operators is modified when changing the resolution in models with non-stretched geometries. In a uniform geometry, the HD operator writes:

$$\frac{\partial X}{\partial t} = -K \nabla^r X, \tag{1}$$

where ∇ is the horizontal derivative operator, K is the HD coefficient, and r is the order of the HD.
The "strength" of any HD operator in a uniform resolution geometry can be quantified by the inverse of the damping time of the shortest resolved wave (the "strength" is hence denoted by τ_s^{-1} hereafter). Examination of various uniform resolution models in operation shows that when changing the resolution (i.e. Δx to fix ideas), τ_s is neither chosen proportional to Δx^r, nor independent of Δx, but rather proportional to Δx. This is due to the fact that the first (resp. second) choice is found to result into a lack (resp. an excess) of activity during the course of long integrations, especially for the smallest resolved scales. As a consequence, the HD can be written:

$$\frac{\partial X}{\partial t} = -k \Delta x^{r-1} \nabla^r X, \tag{2}$$

where the parameter k is independent of the resolution. This empirical result is thus taken as a basis to define an ideal HD operator when the resolution is not uniform, due to a stretched geometry.

3 Ideal HD operator for stretched geometries

When a stretched geometry is used, the actual horizontal coordinate (e.g. x) is replaced by a transformed coordinate $x' = f(x)$. The local map factor is then defined by $\mathrm{m}(x) = df/dx$. The physical gradient operator $\nabla = (\partial/\partial x)$ is thus related to the transformed gradient operator $\nabla' = (\partial/\partial x')$ by: $\nabla = \mathrm{m}\nabla'$. Let us call Δx_0 the mesh at a location where $\mathrm{m} = 1$; we have $\Delta x = \Delta x_0/\mathrm{m}$.
Considering the empirical observation of the previous section, the "ideal" HD operator with a stretched geometry, is the one which has everywhere the same properties as for a non-stretched geometry with the same local resolution. Hence, the ideal HD operator must write:

$$\frac{\partial X}{\partial t} = -k\, \Delta x^{r-1}\, \nabla^r X = -k \Delta x_0^{r-1}\, \mathrm{m}^{1-r}\, \nabla^r X. \tag{3}$$

Generally, the HD operator of the stretched model is rather specified in terms of the transformed derivative operator ∇':

$$\frac{\partial X}{\partial t} = -k\Delta x_0^{r-1}\, \mathrm{m}\nabla'^{r} X. \tag{4}$$

It is worth noting that in (3) and (4), the parameter k is by nature independent of the space, but also of the absolute resolution Δx_0^{-1}. As a consequence, for a given transformation f, k does not need to be changed when increasing the resolution of the model. The parameter k is said space- and resolution-independent.

4 Particular case of Schmidt transformation

The stretched ARPEGE model uses a particular analytical stretching transformation proposed by Schmidt (1977) which allows an algebraic treatment of the equations in the stretched geometry, coupled with the spectral method. Let θ be the latitude on the non-stretched sphere, and Θ the latitude on the stretched sphere. We note $\xi = \sin\Theta$. The ARPEGE stretching transformation is defined by:

$$\theta(\xi) = \arcsin\left(\frac{a\xi + b}{a + b\xi}\right), \tag{5}$$

where $a = (c^2+1)/2c$ and $b = (c^2-1)/2c$ and c is the stretching factor. The local map factor is given by:

$$\mathrm{m}(\xi) = \frac{\partial\Theta}{\partial\theta} = a + b\xi. \tag{6}$$

The original HD operator of ARPEGE (see YB95) consisted in a mixture of a pure ∇'^{r} operator and an approximated $\nabla^u = \mathrm{m}^u\nabla'^{u}$ operator:

$$\frac{\partial X}{\partial t} = -K\mathrm{m}\nabla'^{r}X - K_u\widehat{\mathrm{m}^u}\nabla'^{u}X, \tag{7}$$

where $\widehat{\mathrm{m}^u}$ is a second-degree approximation of m^u given by:

$$\widehat{\mathrm{m}^u} = a_0 + a_1\xi + a_2\xi^2. \tag{8}$$

It appears clearly from (7) and (4) that doing $K = 0$, $u = r$, $a_0 = a$, $a_1 = b$ and $a_2 = 0$, the original ARPEGE HD operator can be transformed into the ideal HD operator defined in the previous section. The advantages of this approach are multiple: (i) the HD operator becomes tri-diagonal in the spectral space instead of penta-diagonal (see YB95); (ii) the obtained HD operator is exactly the ideal one, instead of an approximation of it; (iii) by writing $K_u = k\Delta x_0^{r-1}$ the obtained HD coefficient k can be shown to be space-, resolution-, and stretching-independent. This latter property is important since any change in the geometry can be done without retuning the HD coefficient k, which was not true with the original ARPEGE HD operator defined in YB95.

5 Conclusion

The work presented here allows to rationalize the formulation of HD operators for models with stretched horizontal geometries. For any of these models, an ideal HD operator can be found, allowing the definition of a space- and resolution-independent parameter k. For the special case of the Schmidt stretching transformation, the algebraic nature of the transformation makes it possible to obtain a HD coefficient which is additionally stretching-independent. This new HD operator has been implemented operationally in ARPEGE since february 2003.

References

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